

# Math 2010 Week 9

Application of chain rule:

Level Set (Au 4.4, Thomas 14.6)

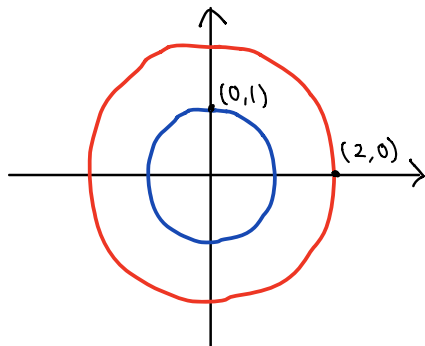
Recall: Let  $f: \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ ,  $c \in \mathbb{R}$

$$L_c = f^{-1}(c) = \{x \in \Omega; f(x) = c\}$$

eg Some level sets of  $f(x,y) = x^2 + y^2$

$$f^{-1}(1) = \{x^2 + y^2 = 1\}$$

$$f^{-1}(4) = \{x^2 + y^2 = 4\}$$



Thm Let  $f: \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ ,  $\Omega$  is open,

let  $c \in \mathbb{R}$ ,  $S = f^{-1}(c)$  and  $a \in S$

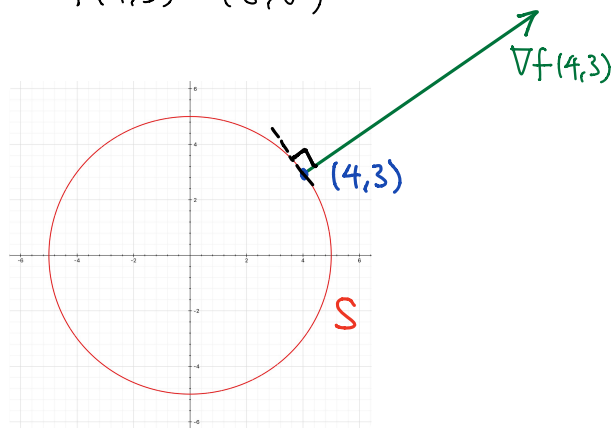
Suppose  $f$  is differentiable at  $a$ ,  $\nabla f(a) \neq 0$

Then  $\nabla f(a) \perp S$  at  $a$

eg  $f(x,y) = x^2 + y^2$      $\nabla f = (2x, 2y)$

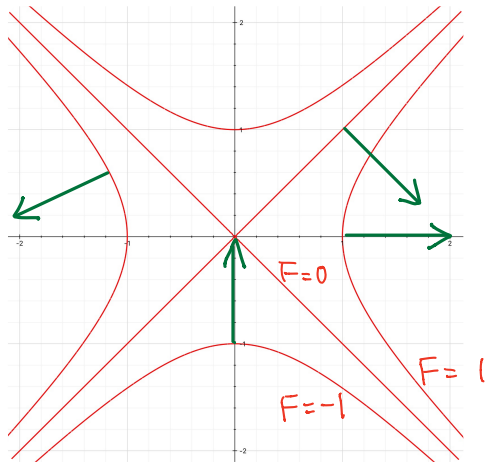
Let  $S = f^{-1}(25)$ , then  $(4,3) \in S$

$$\nabla f(4,3) = (8,6)$$



eg  $f(x,y) = x^2 - y^2$

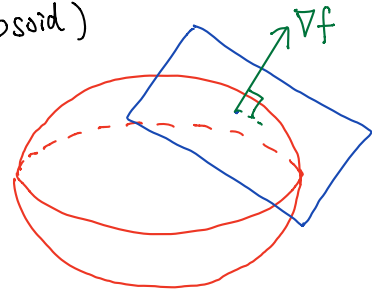
$\nabla f(x,y) = (2x, -2y)$



→ are directions of  $\nabla f$

eg  $S: x^2 + 4y^2 + 9z^2 = 22$  (Ellipsoid)

Find equation of tangent plane of  $S$  at  $(3,1,1)$



Sol

Let  $f(x,y,z) = x^2 + 4y^2 + 9z^2$ ,  $S = f^{-1}(22)$

Also  $f(3,1,1) = 22$ , so  $(3,1,1) \in S$

$\nabla f = (2x, 8y, 18z)$

$\nabla f(3,1,1) = (6, 8, 18) \perp S$  at  $(3,1,1)$

$\therefore (6, 8, 18)$  is a normal vector for tangent plane

Equation of tangent plane:

$$[(x,y,z) - (3,1,1)] \cdot (6,8,18) = 0$$

$$6(x-3) + 8(y-1) + 18(z-1) = 0$$

$$3x + 4y + 9z = 22$$

Thm Let  $f: \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ ,  $\Omega$  is open,  
 let  $c \in \mathbb{R}$ ,  $S = f^{-1}(c)$  and  $a \in S$   
 Suppose  $f$  is differentiable at  $a$ ,  $\nabla f(a) \neq 0$   
 Then  $\nabla f(a) \perp S$  at  $a$

Pf Let  $\gamma(t)$  be a curve on  $S$ ,  $\gamma(0) = a$

Then  $\gamma(t)$  on  $S = f^{-1}(c)$

$\Rightarrow f(\gamma(t)) = c$  is a constant

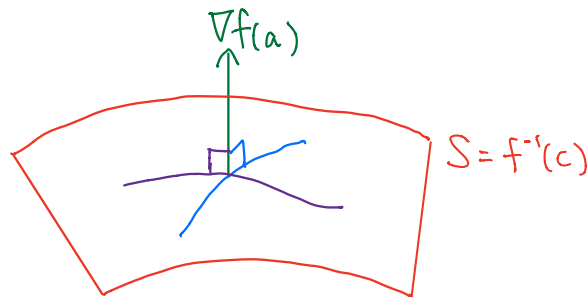
By chain rule,

$$\nabla f(\gamma(t)) \cdot \gamma'(t) = \frac{df}{dt} = 0$$

Put  $t=0$ , then  $\nabla f(a) \cdot \gamma'(0) = 0$

$\therefore \nabla f(a) \perp$  any curve on  $S$  at  $a$ .

$\therefore \nabla f(a) \perp S$  at  $a$



Rmk

We applied chain rule to  $f(\gamma(t))$  above.

Similarly, by considering the curve

$$\alpha(t) = a + t u, \quad \|\vec{u}\| = 1,$$

one can prove

$$D_u f(a) = \left. \frac{d}{dt} \right|_{t=0} f(\alpha(t))$$

$$= \nabla f(\alpha(0)) \cdot \alpha'(0)$$

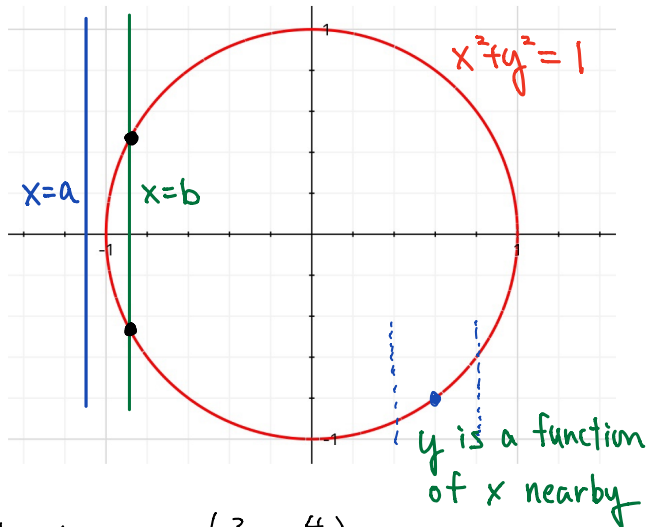
$$= \nabla f(\vec{a}) \cdot u$$

Another application of chain rule:

Implicit differentiation (Au 4.5  
Thomas 14.4)

$$C: x^2 + y^2 = 1$$

$$\text{Find } \frac{dy}{dx} \text{ at } \left(\frac{3}{5}, -\frac{4}{5}\right)$$



Locally near  $\left(\frac{3}{5}, -\frac{4}{5}\right)$ ,

$$y^2 = 1 - x^2, y < 0 \Rightarrow y = -\sqrt{1 - x^2}$$

$\therefore y$  is a function of  $x$  near  $\left(\frac{3}{5}, -\frac{4}{5}\right)$

To find  $\frac{dy}{dx}$  at  $\left(\frac{3}{5}, -\frac{4}{5}\right)$ ,

$$\text{Method 1: } \frac{d}{dx}(-\sqrt{1-x^2})$$

Method 2: Implicit differentiation (chain rule)

$$x^2 + y^2 = 1 \quad \left( \begin{array}{l} \text{Regard } x \text{ as a variable} \\ y \text{ as a function of } x \end{array} \right)$$

$$\text{Take } \frac{d}{dx}: 2x + 2y \frac{dy}{dx} = 0$$

$$\text{Put } (x, y) = \left(\frac{3}{5}, -\frac{4}{5}\right), 2\left(\frac{3}{5}\right) + 2\left(-\frac{4}{5}\right) \frac{dy}{dx} = 0$$

$$\therefore \frac{dy}{dx} \Big|_{\left(\frac{3}{5}, -\frac{4}{5}\right)} = \frac{3}{4}$$

Rmk Near  $(-1, 0)$ ,  $y$  is not a function of  $x$

If  $a < -1$  a little bit, the vertical line  $x=a$  does not intersect with  $C$

If  $b > -1$  a little bit, the vertical line  $x=b$  intersects with  $C$  at two different points



eg Consider

$$S: x^3 + z^2 + ye^{xz} + z \cos y = 0 \quad (*)$$

Given that  $z$  can be regarded as a function

$z = z(x, y)$  of independent variables  $x, y$  locally

near  $(0, 0, 0)$ . Find  $\frac{\partial z}{\partial x}$ ,  $\frac{\partial z}{\partial y}$  at  $(0, 0, 0)$ .

Rmk ① Not easy to express  $z$  in terms of  $x, y$

Sol Take  $\frac{\partial}{\partial x}$  to  $(*)$

$$3x^2 + 2z \frac{\partial z}{\partial x} + ye^{xz} (z + x \frac{\partial z}{\partial x}) + \frac{\partial z}{\partial x} \cos y = 0$$

Put  $(x, y, z) = (0, 0, 0)$ ,

$$0 + 0 + 0 + \frac{\partial z}{\partial x} (1) = 0 \Rightarrow \frac{\partial z}{\partial x} (0, 0) = 0$$

Similarly, take  $\frac{\partial}{\partial y}$  to  $(*)$

$$0 + 2z \frac{\partial z}{\partial y} + e^{xz} + ye^{xz} (x \frac{\partial z}{\partial y}) + \frac{\partial z}{\partial y} \cos y - z \sin y = 0$$

Put  $(x, y, z) = (0, 0, 0)$ , then

$$0 + 0 + 1 + 0 + \frac{\partial z}{\partial y} (1) - 0 = 0$$

$$\Rightarrow \frac{\partial z}{\partial y} (0, 0) = -1$$

Rmk ② From computations above,

$$\frac{\partial z}{\partial x} = - \frac{3x^2 + yze^{xz}}{2z + xye^{xz} + \cos y}$$

$$\frac{\partial z}{\partial y} = \frac{z \sin y - e^{xz}}{2z + xye^{xz} + \cos y}$$

whenever the denominator is non-zero

## Finding Extrema (Maximum & Minimum) (Au ch 5 Thomas 14.7)

Defn Let  $f: A \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ ,  $a \in A$

①  $f$  is said to have global (absolute) maximum at  $a$

if  $f(x) \leq f(a)$  for all  $x \in A$

②  $f$  is said to have local (relative) maximum at  $a$

if  $f(x) \leq f(a)$  for all  $x \in A$  near  $a$ ,

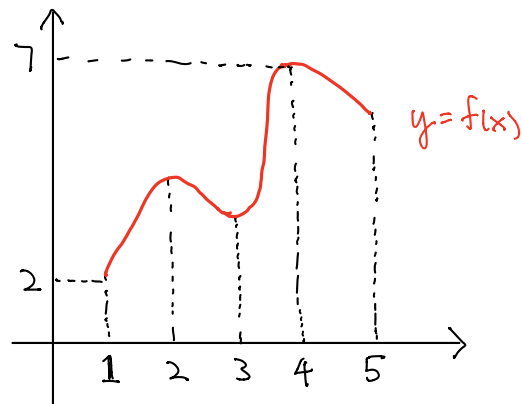
(i.e.  $\exists \varepsilon > 0$  such that  $f(x) \leq f(a)$   
for all  $x \in A \cap D_\varepsilon(a)$ )

③ Similar definitions for global (absolute) minimum

and local (relative) minimum

Rmk Global extremum (max/min) is also  
a local extremum.

eg let  $f: [1, 5] \rightarrow \mathbb{R}$



Global max at 4

Global min at 1

Local max. at 2, 4

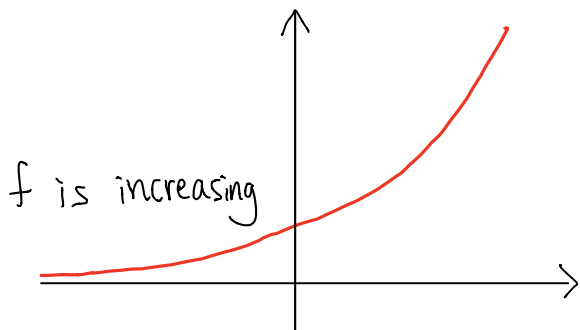
Local min. at 1, 3, 5

Max Value : 7

Min Value : 2

Rmk NOT every function has  
global max/min!

①  $f(x) = e^x$  on  $\mathbb{R}$



$$\lim_{x \rightarrow -\infty} f(x) = 0$$

$$\lim_{x \rightarrow \infty} f(x) = \infty$$

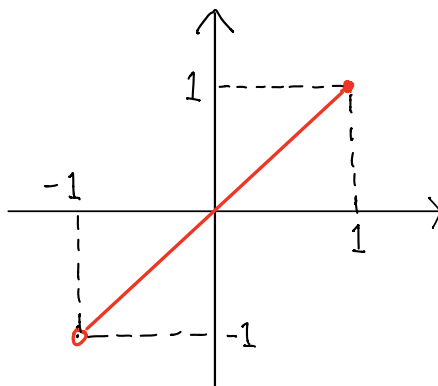
But  $f(x) > 0 \forall x \in \mathbb{R}$

$\therefore$  No global max

$\therefore$  No global min

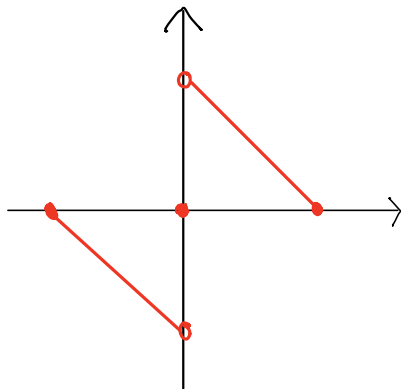
(Domain is not bounded)

②  $f(x) = x$  on  $(-1, 1]$  (Domain is not closed)



$f$  has global max at 1  
but no global min

③  $f: [-1, 1] \rightarrow \mathbb{R}$   $f(x) = \begin{cases} 1-x & \text{if } x \in (0, 1] \\ 0 & \text{if } x = 0 \\ -1-x & \text{if } x \in [-1, 0) \end{cases}$



$f$  has neither  
global max or min

( $f$  is not continuous)

Q When must a function have global max/min?

### Extreme Value Theorem (EVT)

Let  $A \subseteq \mathbb{R}^n$  be closed and bounded

$f: A \rightarrow \mathbb{R}$  is continuous

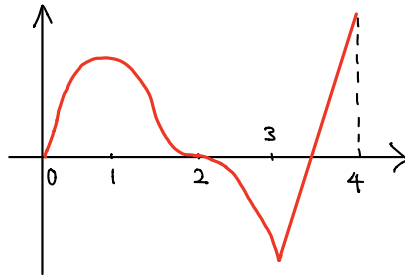
Then  $f$  has global max and min.

Rmk

- ① Compact = closed and bounded
- ② The theorem provides a sufficient condition, but not necessary condition for the existence of global extrema

Q How to locate max/min?

eg  $f: A = [0, 4] \rightarrow \mathbb{R}$



$A$  is closed, bounded

$f$  is continuous

$\Rightarrow f$  has global max and min

Recall One variable calculus:

Extrema can only occur at

- $f'(x) = 0 : x = 1, 2$
  - $f'(x) \text{ DNE} : x = 3$
  - $x \in \partial A : x = 0, 4$
- Critical points
- Boundary points

Comparing values of  $f$  at these 5 points:

$f$  has global min at 3, global max at 4.

Defn  $f: A \rightarrow \mathbb{R}$ ,  $a \in \text{int}(A)$ .

Then  $a$  is called a critical point of  $f$  if either

①  $\nabla f(a)$  DNE (i.e.  $\frac{\partial f}{\partial x_i}(a)$  DNE for some  $i$ )

②  $\nabla f(a) = \vec{0}$  (i.e.  $\frac{\partial f}{\partial x_i}(a) = 0$  for all  $i$ )

Thm (First derivative test)

Suppose  $f: A \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$  attains a local extremum at  $a \in \text{int}(A)$ , then  $a$  is a critical point of  $f$

Pf Suppose  $f$  has a local extremum at  $a \in \text{int}(A)$

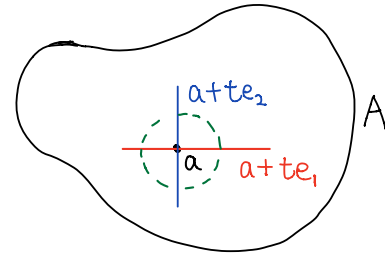
If  $\nabla f(a)$  DNE, then  $a$  is a critical point.

If  $\nabla f(a)$  exists, then all  $\frac{\partial f}{\partial x_i}(a)$  exist.

For any  $i = 1, \dots, n$ , let

$$g_i(t) = f(a + te_i)$$

Note  $a \in \text{int}(A) \Rightarrow g_i(t)$  is defined near  $t=0$



By definition,  $g_i'(0) = \frac{\partial f}{\partial x_i}(a)$  exists.

$f$  has local extremum at  $a$

$\Rightarrow g_i$  has local extremum at  $0$

$\Rightarrow g_i'(0) = 0$  since it exists

$\Rightarrow \frac{\partial f}{\partial x_i}(a) = 0$  (for any  $i = 1, 2, \dots, n$ )

$\therefore \nabla f(a) = \vec{0}$

$\therefore a$  is a critical point.

## Strategy for finding extrema

Let  $f: A \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ .

① Find critical points of  $f$  in  $\text{int}(A)$

② Study  $f$  on boundary  $\partial A$ :

Find max/min of  $f$  on  $\partial A$

③ Comparing values of  $f$  at points found in ① and ②

eg Find global max/min of

$$f(x,y) = x^2 + 2y^2 - x + 3 \text{ for } x^2 + y^2 \leq 1$$

Rmk Domain =  $A = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$

is closed and bounded

Also,  $f$  is polynomial  $\Rightarrow f$  is continuous

By EVT,  $f$  has global max and min on  $A$

Sol We follow the strategy above:

Step 1: Critical points in  $\text{int}(A)$

$\nabla f = (2x-1, 4y)$  exists everywhere

$$\text{Also, } \nabla f = \vec{0} \Leftrightarrow \begin{cases} 2x-1=0 \\ 4y=0 \end{cases}$$

$$\Leftrightarrow (x,y) = \left(\frac{1}{2}, 0\right)$$

Note  $\left(\frac{1}{2}, 0\right) \in \text{int}(A) = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$

$\therefore f$  has only one critical point  $\left(\frac{1}{2}, 0\right)$  in  $\text{int}(A)$  with

$$f\left(\frac{1}{2}, 0\right) = \left(\frac{1}{2}\right)^2 + 0 - \frac{1}{2} + 3 = \frac{11}{4}$$

Step 2: Study  $f$  on  $\partial A = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$

Parametrize  $\partial A$  by

$$(x,y) = (\cos \theta, \sin \theta), \theta \in [0, 2\pi]$$

$$f(\cos \theta, \sin \theta) = \cos^2 \theta + 2\sin^2 \theta - \cos \theta + 3$$

$$= \cos^2 \theta + 2(1 - \cos^2 \theta) - \cos \theta + 3$$

$$= -\cos^2 \theta - \cos \theta + 5$$

$$= -\left(\cos \theta + \frac{1}{2}\right)^2 + \frac{1}{4} + 5$$

$$= \frac{21}{4} - \left(\cos \theta + \frac{1}{2}\right)^2$$

max value of  $f$  on  $\partial A = \frac{21}{4}$ , occurs when

$$x = \cos \theta = -\frac{1}{2}, \text{ i.e. } (x,y) = \left(-\frac{1}{2}, \pm \frac{\sqrt{3}}{2}\right)$$

min value of  $f$  on  $\partial A = 3$  occurs when

$$x = \cos \theta = 1, \text{ i.e. } (x,y) = (1, 0)$$

Step 3: Compare values of  $f$  at points from step 1 and 2.

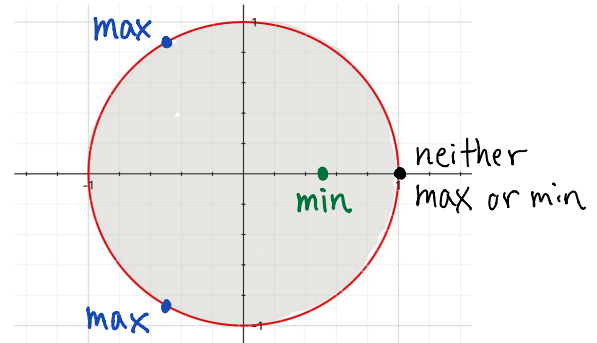
$$f\left(\frac{1}{2}, 0\right) = \frac{11}{4} \leftarrow \text{min}$$

$$f\left(-\frac{1}{2}, \pm \frac{\sqrt{3}}{2}\right) = \frac{21}{4} \leftarrow \text{max}$$

$$f(1, 0) = 3$$

Max value =  $\frac{21}{4}$  at  $\left(-\frac{1}{2}, \pm \frac{\sqrt{3}}{2}\right)$

Min value =  $\frac{11}{4}$  at  $\left(\frac{1}{2}, 0\right)$



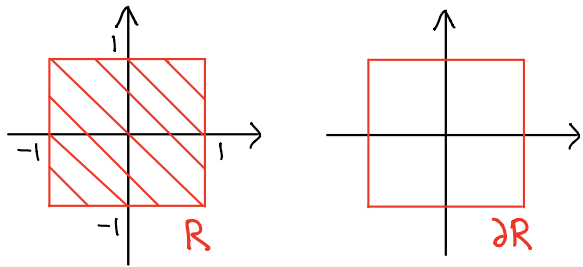
eg 2 Find global max and min of

$$f(x,y) = \sqrt{x^2+y^4} - y$$

$$\text{on } R = \{(x,y) \in \mathbb{R}^2 : -1 \leq x,y \leq 1\}$$

Sol

$R$  is the square  $[-1,1] \times [-1,1] \subseteq \mathbb{R}^2$



$R$  is closed and bounded

Also,  $f$  is continuous

By EVT,  $f$  has global max and min.

To find them...

Step 1 Study  $\text{int}(R) = \{(x,y) \in \mathbb{R}^2, -1 < x,y < 1\}$

Ex Show that  $\frac{\partial f}{\partial x}(0,0)$  DNE ( $f(x,0) = |x|$ )

For  $(x,y) \neq (0,0)$ ,  $\nabla f$  exists and

$$\nabla f = \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right) = \left( \frac{x}{\sqrt{x^2+y^4}}, \frac{2y^3}{\sqrt{x^2+y^4}} - 1 \right)$$

$$\nabla f = (0,0) \Leftrightarrow \begin{cases} \frac{x}{\sqrt{x^2+y^4}} = 0 \\ \frac{2y^3}{\sqrt{x^2+y^4}} - 1 = 0 \end{cases}$$

$$\therefore x=0, \frac{2y^3}{y^2} - 1 = 0 \Rightarrow y = \frac{1}{2}$$

$\therefore f$  has two critical points in  $\text{int}(A)$

$$(0,0) \quad \text{and} \quad (0, \frac{1}{2})$$

$\nabla f$  DNE

$$\nabla f = \vec{0}$$

$$f(0,0) = 0, \quad f(0, \frac{1}{2}) = -\frac{1}{4}$$

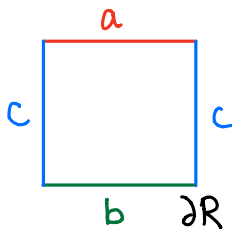


## Step 2 Study $f$ on $\partial R$

$$f(x,y) = \sqrt{x^2+y^4} - y$$

$$\partial R = \{(x,y) : |x|=1, -1 \leq y \leq 1\}$$

$$\cup \{(x,y) : |y|=1, -1 \leq x \leq 1\}$$



Consider different parts of  $\partial R$ :

(a)  $y=1, -1 \leq x \leq 1$

$$f(x,1) = \sqrt{x^2+1} - 1 \Rightarrow 0 \leq f \leq \sqrt{2} - 1$$

(b)  $y=-1, -1 \leq x \leq 1$

$$f(x,-1) = \sqrt{x^2+1} + 1 \Rightarrow 2 \leq f \leq \sqrt{2} + 1$$

(c)  $|x|=1, -1 \leq y \leq 1$  Not the sharpest bound but good enough for us \*

$$f(x,y) = \sqrt{1+y^4} - y \Rightarrow 0 = 1 - 1 \leq f \leq \sqrt{2} + 1$$

$\therefore$  On  $\partial R$ ,  $f$  has min value 0 at  $(0,1)$

max value  $\sqrt{2}+1$  at  $(\pm 1, -1)$

## Step 3 Comparing values

$$f(0,0) = 0$$

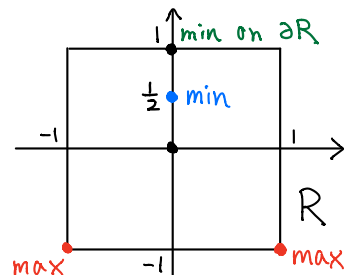
$$f(0, \frac{1}{2}) = -\frac{1}{4} \leftarrow \text{min}$$

$$f(0,1) = 0$$

$$f(\pm 1, -1) = \sqrt{2} + 1 \leftarrow \text{max}$$

Max value =  $\sqrt{2} + 1$  at  $(\pm 1, -1)$

Min value =  $-\frac{1}{4}$  at  $(0, \frac{1}{2})$



\* We may find the sharpest bound.

By one variable calculus, easy to see

$\sqrt{1+y^4} - y$  is decreasing on  $[-1, 1]$

$\Rightarrow$  min value on (c) =  $\sqrt{1+1^2} - 1 = \sqrt{2} - 1$

## Finding extrema on an unbounded region

eg Find global extrema of

$$f(x,y) = x^2 + y^2 - 4x + 6y + 7 \text{ on } \mathbb{R}^2$$

Rmk  $\mathbb{R}^2$  is not bounded. So  $f$  may not have global extrema. Observe that

$$\lim_{\underbrace{(x,y) \rightarrow \infty}} f(x,y) = +\infty. \text{ Hence}$$

" $(x,y)$  are far away from origin"

①  $f$  has no global maximum on  $\mathbb{R}^2$

② Strategy for finding global minimum:

Find a closed and bounded region  $R$  s.t.

$f$  is "large enough" outside  $R$ . Then

$$\min \text{ on } R = \min \text{ on } \mathbb{R}^2$$

Sol Find critical points of  $f$ :

$$\nabla f = \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right) = (2x-4, 2y+6) \text{ exists on } \mathbb{R}^2$$

$$\nabla f = (0,0) \Leftrightarrow \begin{cases} 2x-4=0 \\ 2y+6=0 \end{cases}$$

$$\Leftrightarrow (x,y) = (2, -3)$$

$\therefore$  Only one critical point  $(2, -3)$ ,  $f(2, -3) = -6$

We want to show  $f$  has min at  $(2, -3)$ :

$$\text{For } (x,y) \in \mathbb{R}^2, \text{ let } r = \sqrt{x^2 + y^2}$$

$$\text{Then } f(x,y) = x^2 + y^2 - 4x + 6y + 7$$

$$\textcircled{*} \geq r^2 - 4r - 6r + 7$$

$$= r(r-10) + 7$$

$$\textcircled{*} \quad r = \sqrt{x^2 + y^2} \geq |x|, |y| \Rightarrow \begin{cases} x \leq r \Rightarrow -4x \geq -4r \\ -y \leq r \Rightarrow 6y \geq -6r \end{cases}$$

Hence, if  $\sqrt{x^2+y^2} = r \geq 10$

then  $f(x,y) \geq 7 > f(2,-3)$

Let  $R = \overline{D_{10}(0,0)}$ ,  $f|_R =$  restriction of  $f$  on  $R$

By EVT,  $f|_R$  has global min.

In  $\text{Int}(R)$ ,  $(2,-3)$  is the only critical point

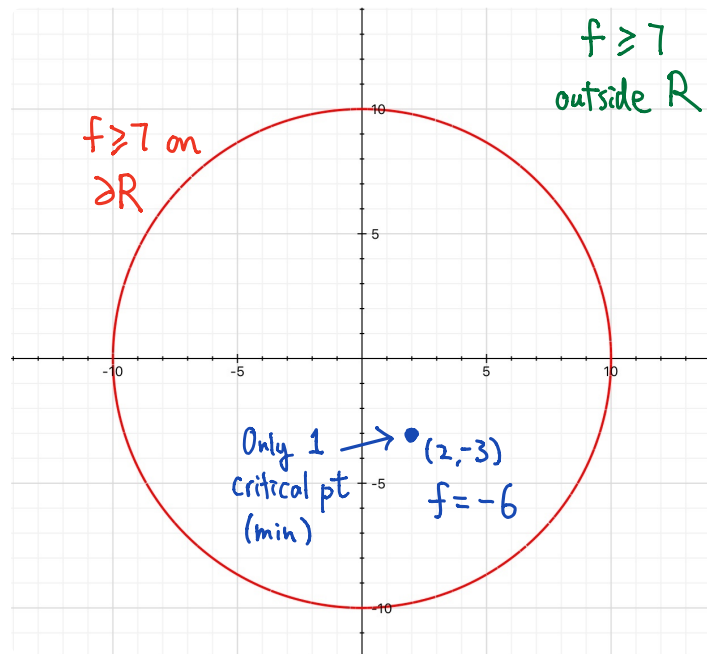
$$f(2,-3) = -6$$

On  $\partial R$ ,  $f(x,y) \geq 7 > f(2,-3)$

$\therefore f|_R$  has global min at  $(2,-3)$

For  $(x,y) \notin R$ ,  $f(x,y) \geq 7 > f(2,-3)$

$\therefore f$  has no global max and global min. value  $-6$  at  $(2,-3)$



Rmk

① Easier to do this question using algebra.

$$\begin{aligned} f(x,y) &= x^2 + y^2 - 4x + 6y + 7 \\ &= (x-2)^2 + (y+3)^2 - 6 \end{aligned}$$

Answer is clear

② An example without global max or min.

$$\text{Let } g(x,y) = x^2 - y^2 - 4x + 6y + 7$$

$$\text{When } x=0, g(0,y) = -y^2 + 6y + 7$$

$$\lim_{y \rightarrow \pm\infty} g(0,y) = -\infty \Rightarrow \text{no global min.}$$

$$\text{When } y=0, g(x,0) = x^2 - 4x + 7$$

$$\lim_{x \rightarrow \pm\infty} g(x,0) = \infty \Rightarrow \text{no global max}$$